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The Chrono-Geometrical Structure of General Relativity and Clock Synchronization.

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Abstract

After a review of the chrono-geometrical structure of special relativity, where the definition of the instantaneous 3-space is based on the observer-dependent convention for the synchronization of distant clocks, it is shown that in a class of models of general relativity the instantaneous 3-space and the associated clock synchronization convention are dynamically determined by Einstein's equations. This theoretical framework is necessary to understand the relativistic effects around the Earth, to be tested with the ACES mission of ESA, and the implications for metrology induced by the accuracy of the new generation of atomic clocks.

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In Newton physics there are distinct absolute notions of *time* and *space*, so that we can speak of absolute simultaneity and of instantaneous Euclidean 3-spaces with the associated Euclidean spatial distance notion. This non-dynamical chrono-geometrical structure is formalized in the so called Galilei space-time. The Galilei relativity principle assumes the existence of preferred inertial frames with inertial Cartesian coordinates centered on inertial observers, connected by the kinematical group of Galilei transformations. In Newton gravity the equivalence principle states the equality of inertial and gravitational mass. In non-inertial frames inertial (or fictitious) forces proportional to the mass of the body appear in Newton's equations.

In special relativity the structure of the light-cones is an absolute non-dynamical object [1]: they are the only information (the conformal structure) given by the theory to an (either inertial or accelerated) observer in each point of her/his world-line. There is no notion of instantaneous 3-space, of spatial distance and of one-way velocity of light between two observers. The light postulates say that the two-way (or round trip) velocity of light c (only one clock is needed in its definition) is constant and isotropic. For an ideal inertial observer Einstein's convention for the synchronization of distant clocks¹ selects the constant time hyper-planes of the inertial frame having the observer as time axis as the instantaneous Euclidean 3-spaces, with their Euclidean 3-geodesic spatial distance and with the one-way and two-way velocities of light equal.

But this convention does not work for realistic accelerated observers, because coordinate singularities are produced in the attempt (the 1+3 point of view) to build (either Fermi or rotating) 4-coordinates around the observer world-line. They must use the more complex conventions arising from the introduction of an extra structure: a global 3+1 splitting of Minkowski space-time (a choice of *time*, the starting point of the Hamiltonian formalism). Each space-like leaf of the associated foliation is *both a Cauchy surface for the field equations and a convention* (different from Einstein's one) *for clock synchronization*. If we introduce *Lorentz-scalar observer-dependent radar 4-coordinates* $x^\mu \mapsto \sigma^A = (\tau; \sigma^r)$, where x^μ are Cartesian coordinates, τ is an arbitrary monotonically increasing function of the proper time of the observer and σ^r curvilinear 3-coordinates having the observer world-line as origin, this leads to the definition of a *non-inertial frame* centered on the accelerated observer [2]. Every 3+1 splitting, satisfying certain Møller restrictions (to avoid coordinate singularities) and with the leaves tending to hyper-planes at spatial infinity (so that there are asymptotic inertial observers to be identified with the fixed stars), gives a conventional definition of instantaneous 3-space (in general a Riemannian 3-manifold), of 3-geodesic spatial distance and of one-way velocity of light (in general both point-dependent and anisotropic). The inverse coordinate transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* of the simultaneity surfaces Σ_τ into Minkowski space-time. The 3+1 splitting leads to the following induced 4-metric (a functional of the embedding): ${}^4g_{AB}(\tau, \sigma^r) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} {}^4\eta_{\mu\nu} \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} = {}^4g_{AB}[z(\sigma)]$.

Parametrized Minkowski theories [3], [1] allow to give a description of every isolated system (particles, strings, fields, fluids), in which the transition from a 3+1 splitting to another one (i.e. a change of clock synchronization convention) is a *gauge transfor-*

¹ The inertial observer A sends a ray of light at x_i^o to a second accelerated observer B , who reflects it towards A . The reflected ray is reabsorbed by the inertial observer at x_f^o . The convention states that the clock of B at the reflection point must be synchronized with the clock of A when it signs $\frac{1}{2}(x_i^o + x_f^o) = x_i^o + \frac{1}{2}(x_f^o - x_i^o)$.

mation. Given any isolated system admitting a Lagrangian description, one makes the coupling of the system to an external gravitational field and then replaces the 4-metric ${}^4g_{\mu\nu}(x)$ with the induced metric ${}^4g_{AB}[z(\tau, \sigma^r)]$ associated to an arbitrary admissible 3+1 splitting. The Lagrangian now depends not only on the matter configurational variables but also on the embedding variables $z^\mu(\tau, \sigma^r)$ (whose conjugate canonical momenta are denoted $\rho_\mu(\tau, \sigma^r)$). Since the action principle turns out to be invariant under *frame-preserving diffeomorphisms*, at the Hamiltonian level there are four first-class constraints $\mathcal{H}_\mu(\tau, \sigma^r) = \rho_\mu(\tau, \sigma^r) - l_\mu(\tau, \sigma^r) T^{\tau\tau}(\tau, \sigma^r) - z^\mu_s(\tau, \sigma^r) T^{\tau s}(\tau, \sigma^r) \approx 0$ in strong involution with respect to Poisson brackets, $\{\mathcal{H}_\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma^r_1)\} = 0$. Here $l_\mu(\tau, \sigma^r)$ are the covariant components of the unit normal to Σ_τ , while $z^\mu_s(\tau, \sigma^r) = \frac{\partial z^\mu(\tau, \sigma^r)}{\partial \sigma^s}$ are the components of three independent vectors tangent to Σ_τ . The quantities $T^{\tau\tau}$ and $T^{\tau s}$ are the components of the energy-momentum tensor of the matter inside Σ_τ describing its energy- and momentum-densities. Since the first class constraints are generators of Hamiltonian gauge transformations, this implies that the configuration variables $z^\mu(\tau, \sigma^r)$ are arbitrary *gauge variables*. Therefore, all the admissible 3+1 splittings, namely all the admissible conventions for clock synchronization, and all the admissible non-inertial frames centered on time-like observers are *gauge equivalent*.

By adding four gauge-fixing constraints $\chi^\mu(\tau, \sigma^r) = z^\mu(\tau, \sigma^r) - z^\mu_M(\tau, \sigma^r) \approx 0$ ($z^\mu_M(\tau, \sigma^r)$ being an admissible embedding), satisfying the orbit condition $\det |\{\chi^\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma^r_1)\}| \neq 0$, we identify the description of the system in the associated non-inertial frame centered on a given time-like observer. The resulting effective Hamiltonian for the τ -evolution turns out to contain the potentials of the *relativistic inertial forces* present in the given non-inertial frame. As a consequence, the gauge variables $z^\mu(\tau, \sigma^r)$ describe the *spatio-temporal appearances* of the phenomena in non-inertial frames, which, in turn, are associated to *extended* physical laboratories using a metrology for their measurements compatible with the notion of simultaneity (distant clock synchronization convention) of the non-inertial frame (think to the description of the Earth given by GPS). Therefore, notwithstanding mathematics tends to use only coordinate-independent notions, physical metrology forces us to consider intrinsically coordinate-dependent quantities like the non-inertial Hamiltonians. For instance, the motion of satellites around the Earth is governed by a set of empirical coordinates contained in the software of NASA computers [4]: this is a *metrological standard of space-time around the Earth*.

Inertial frames centered on inertial observers are a special case of gauge fixing in parametrized Minkowski theories. For each configuration of an isolated system there is an special 3+1 splitting associated to it: the foliation with space-like hyper-planes orthogonal to the conserved time-like 4-momentum of the isolated system. This identifies an intrinsic inertial frame, the *rest-frame*, centered on a suitable inertial observer (the covariant non-canonical Fokker-Pryce center of inertia of the isolated system) and allows to define the *Wigner-covariant inertial rest-frame instant form of dynamics* for every isolated system, which allows to give a new formulation of the relativistic kinematics [5] of N-body systems and continuous media (relativistic centers of mass and canonical relative variables, rotational kinematics and dynamical body frames, multipolar expansions, Møller radius) and to find the theory of relativistic orbits. Instead *non-inertial rest frames* are 3+1 splittings of Minkowski space-time having the associated simultaneity 3-surfaces tending to Wigner hyper-planes at spatial infinity.

Instead in general relativity there is no absolute notion [1]: the full chrono-geometrical

structure of space-time is dynamical. The relativistic description of gravity abandons the relativity principle and replaces it with the equivalence principle. Special relativity can be recovered only locally by a freely falling observer in a neighborhood where tidal effects are negligible. As a consequence, *global inertial frames do not exist*.

The general covariance of Einstein's formulation of general relativity leads to a type of gauge symmetry acting also on space-time: the Hilbert action is invariant under coordinate transformations (*passive* off-shell diffeomorphisms as local Noether transformations). As in parametrized Minkowski theories the gauge variables are arbitrary degrees of freedom connected with the *appearances* of phenomena in the various coordinate systems of Einstein's space-times.

In Einstein's geometrical view of the gravitational field the basic configuration variable is the metric tensor over space-time (10 fields), which, differently from every other field, has a double role:

- i) it is the mediator of the gravitational interaction, like every other gauge field;
- ii) it describes the dynamical chrono-geometrical structure of space-time by means of the line element $ds^2 = {}^4g_{\mu\nu}(x) dx^\mu dx^\nu$. As a consequence, it *teaches relativistic causality* to the other fields: now the conformal structure (the allowed paths of light rays) is point-dependent.

In canonical ADM metric gravity [6] (and in its tetrad gravity extension [7] needed for fermions ²) we have again to start with the same pattern of 3+1 splittings, to be able to define the Cauchy and simultaneity surfaces for Einstein's equations. As a consequence, and having in mind the inclusion of particle physics, we must select a family of *non-compact* space-times M^4 with the following properties:

- i) *globally hyperbolic* and *topologically trivial*, so that they can be foliated with space-like hyper-surfaces Σ_τ diffeomorphic to R^3 ;
- ii) *asymptotically flat at spatial infinity* and with boundary conditions at spatial infinity independent from the direction, so that the *spi group* of asymptotic symmetries is reduced to the Poincare' group with the ADM Poincare' charges as generators. In this way we can eliminate the *super-translations*, namely the obstruction to define angular momentum in general relativity. All these requirements imply that the *admissible foliations* of space-time must have the space-like hyper-surfaces tending in a direction-independent way to Minkowski space-like hyper-planes at spatial infinity, which moreover must be orthogonal there to the ADM 4-momentum. Therefore, M^4 is *asymptotically Minkowskian* [8] with the asymptotic Minkowski metric playing the role of an *asymptotic background*. In absence of matter the class of Christodoulou-Klainermann space-times [9], admitting asymptotic ADM Poincare' charges and an asymptotic flat metric meets these requirements.

This formulation, the *rest-frame instant form of metric and tetrad gravity*, emphasizes the role of *non-inertial frames* (the only ones existing in general relativity): each admissible 3+1 splitting identifies a global non-inertial frame centered on a time-like observer. In these space-times each simultaneity surface is the rest frame of the 3-universe, there are asymptotic inertial observers (the fixed stars) and the switching off of the Newton constant in presence of

² This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer's gyroscopes.

matter leads to a deparametrization of these models of general relativity to the non-inertial rest-frame instant form of the same matter with the ADM Poincare' charges collapsing into the usual kinematical Poincare' generators. This class of space-times is suitable to describe the solar system (or the galaxy), is compatible with particle physics and allows to avoid the splitting of the metric into a background one plus a perturbation. With the addition of suitable asymptotic terms it can probably be adapted to cosmology [10].

The first-class constraints of canonical gravity (8 in metric gravity, 14 in tetrad gravity ³) imply the existence of an equal number of arbitrary gauge variables and of only 2+2 genuine physical degrees of freedom of the gravitational field: $r_{\bar{a}}(\tau, \sigma^r)$, $\pi_{\bar{a}}(\tau, \sigma^r)$. It can be shown [6, 7, 11] that the super-hamiltonian constraint generates Hamiltonian gauge transformations implying the *gauge equivalence* of clock synchronization conventions like it happens in special relativity (no Wheeler-DeWitt interpretation of it as a Hamiltonian). As shown in Refs.[11], the gauge variables describe *generalized inertial effects* (the appearances), while the 2+2 gauge invariant DO describe *generalized tidal effects*. In Refs.[7] a canonical basis, adapted to 13 of the 14 tetrad gravity first class constraints (not to the super-Hamiltonian one) was found. With its help it can be shown [11] that a completely fixed Hamiltonian gauge is equivalent to the choice of a *non-inertial frame* with its adapted radar coordinates centered on an accelerated observer and its instantaneous 3-spaces (simultaneity surfaces): again this corresponds to an extended physical laboratory ⁴.

In the rest-frame instant form of gravity [6, 7], due to the DeWitt surface term the effective Hamiltonian is not weakly zero (*no frozen picture* of dynamics), but is given by the weak ADM energy $E_{ADM} = \int d^3\sigma \mathcal{E}_{ADM}(\tau, \sigma^r)$ (it is the analogous of the definition of the electric charge as the volume integral of the charge density in electromagnetism). The ADM energy density depends on the gauge variables, namely it is a coordinate-dependent quantity (the *problem of energy* in general relativity). In a completely fixed gauge, in which the inertial effects are given functions of the DO, $\mathcal{E}_{ADM}(\tau, \sigma^r)$ becomes a well defined function only of the DO's and there is a deterministic evolution of the DO's (the tidal effects) given by the Hamilton equations. A universe M^4 (a 4-geometry) is the equivalence class of all the completely fixed gauges with gauge equivalent Cauchy data for the DO on the associated Cauchy and simultaneity surfaces Σ_τ . In each completely fixed gauge (an off-shell non-inertial frame determined by some set of gauge-fixing constraints determining the gauge variables in terms of the tidal ones) we find the solution for the DO in that gauge (the tidal effects) and then the explicit form of the gauge variables (the inertial effects). As a consequence, the final admissible (on-shell gauge equivalent) non-inertial frames associated to a 4-geometry (and

³ Tetrad gravity has 10 primary first class constraints and 4 secondary first class ones. Six of the primary constraints describe the extra freedom in the choice of the tetrads. The other 4 primary (the vanishing of the momenta of the lapse and shift functions) and 4 secondary (the super-Hamiltonian and super-momentum constraints) constraints are the same as in metric gravity.

⁴ Let us remark that, if we look at Minkowski space-time as a special solution of Einstein's equations with $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$ (zero Riemann tensor, no tidal effects, only inertial effects), we find [6] that the dynamically admissible 3+1 splittings (non-inertial frames) must have the simultaneity surfaces Σ_τ *3-conformally flat*, because the conditions $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$ imply the vanishing of the Cotton-York tensor of Σ_τ . Instead, in special relativity, considered as an autonomous theory, all the non-inertial frames compatible with the Møller conditions are admissible [5], namely there is much more freedom in the conventions for clock synchronization.

their instantaneous 3-spaces, i.e. their clock synchronization conventions) are *dynamically determined* [11].

A first application of this formalism has been the determination (see the third paper in Refs.[7]) of *post-Minkowskian background-independent gravitational waves* in a completely fixed non-harmonic 3-orthogonal gauge with diagonal 3-metric by adding the weak field requirement $r_{\bar{a}}(\tau, \sigma^r) \ll 1$, $\pi_{\bar{a}}(\tau, \sigma^r) \ll 1$. We get a solution of linearized Einstein's equations, in which the configurational DO $r_{\bar{a}}(\tau, \sigma^r)$ play the role of the *two polarizations* of the gravitational wave.

However, the point canonical transformation of Refs.[7], adapted to 13 of the 14 first class constraints is not suited for the inclusion of matter due to its *non-locality*. Therefore the search started for a local point transformation adapted only to 10 of the 14 constraints, i.e. not adapted to the super-Hamiltonian and super-momentum constraints.

In Ref.[12] (inspired by the so-called York - Lichnerowicz conformal approach [13] to metric gravity in globally hyperbolic space-times based on the decomposition ${}^3g_{ij} = \phi^4 {}^3\hat{g}_{ij}$ of the 3-metric with $\phi = (det {}^3g)^{1/12}$ being the *conformal factor*) a new parametrization of the original 3-metric ${}^3g_{ij}$ was proposed, which allows to find local point canonical transformation, adapted to 10 of the 14 constraints of tetrad gravity, implementing a York map. The 3-metric ${}^3g_{rs}$ may be diagonalized with an *orthogonal* matrix $V(\theta^r)$, $V^{-1} = V^T$, $det V = 1$, depending on 3 Euler angles θ^r . The gauge Euler angles θ^r give a description of the 3-coordinate systems on Σ_τ from a local point of view, because they give the orientation of the tangents to the 3 coordinate lines through each point (their conjugate momenta $\pi_i^{(\theta)}$ are determined by the super-momentum constraints), ϕ is the conformal factor of the 3-metric, i.e. the unknown in the super-hamiltonian constraint (its conjugate momentum is the gauge variable describing the form of the simultaneity surfaces Σ_τ), while the two independent eigenvalues of the conformal 3-metric ${}^3\hat{g}_{rs}$ (with determinant equal to 1) describe the genuine *tidal* effects $R_{\bar{a}}$, $\bar{a} = 1, 2$, of general relativity (the non-linear "graviton", with conjugate momenta $\Pi_{\bar{a}}$). In the York canonical basis [12] the gauge variable, which describes the freedom in the choice of the clock synchronization convention, i.e. in the definition of the instantaneous 3-spaces Σ_τ , is the trace ${}^3K(\tau, \sigma^r)$ of the extrinsic curvature of Σ_τ .

The tidal effects $R_{\bar{a}}(\tau, \sigma^r)$, $\Pi_{\bar{a}}(\tau, \sigma^r)$, are DO *only* with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ become two pairs of *non canonical* DO for that gauge: the two pairs of canonical DO have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the $R_{\bar{a}}$, $\Pi_{\bar{a}}$ variables. This shows the importance of canonical bases like the York one: the tidal effects are described by *local* functions of the 3-metric and its conjugate momenta.

The *arbitrary gauge variables* of the York canonical basis are $\alpha_{(a)}$, $\varphi_{(a)}$, θ^i , $\pi_{\bar{\phi}}$, n and $\bar{n}_{(a)}$. As shown in Refs.[11, 12], they describe the following generalized *inertial effects*:

a) the angles $\alpha_{(a)}(\tau, \sigma^r)$ and the boost parameters $\varphi_{(a)}(\tau, \sigma^r)$ describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point (τ, σ^r) . They fix *the unit 4-velocity of the observer and the conventions for the gyroscopes and their transport along the world-line of the observer*.

b) the angles $\theta^i(\tau, \sigma^r)$ (depending only on the 3-metric) describe the arbitrariness in the choice of the 3-coordinates on the simultaneity surfaces Σ_τ of the chosen non-inertial frame

centered on an arbitrary time-like observer. Their choice induces a pattern of *relativistic standard inertial forces* (centrifugal, Coriolis,...), whose potentials are contained in the weak ADM energy E_{ADM} . These inertial effects are the relativistic counterpart of the non-relativistic ones (they are present also in the non-inertial frames of Minkowski space-time).

c) the *shift* functions $\bar{n}_{(a)}(\tau, \sigma^r)$, appearing in the Dirac Hamiltonian, describe which points on different simultaneity surfaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components ${}^4g_{\tau r}(\tau, \sigma^r)$ of the 4-metric, namely they are the *gravito-magnetic potentials* responsible of effects like the dragging of inertial frames (Lens-Thirring effect) (see the Ciufolini-Wheeler book in Refs.[13]) in the post-Newtonian approximation.

d) $\pi_\phi(\tau, \sigma^r)$, i.e. the York time ${}^3K(\tau, \sigma^r)$, describes the arbitrariness in the shape of the simultaneity surfaces Σ_τ of the non-inertial frame, namely the arbitrariness in the choice of the convention for the synchronization of distant clocks. Since this variable is present in the Dirac Hamiltonian ($H_D = A_{ADM} + constraints$), it is a *new inertial potential* connected to the problem of the relativistic freedom in the choice of the *instantaneous 3-space*, which has no non-relativistic analogue (in Galilei space-time there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored. For instance, since the sign of the trace of the extrinsic curvature may change from a region to another one on the simultaneity surface Σ_τ , *the associated inertial force in the Hamilton equations may change from attractive to repulsive in different regions* since H_D knows this sign. These inertial forces could imply that part of dark matter is an inertial effect (see Ref.[14] for a possible gravito-magnetic origin).

e) the *lapse* function $N(\tau, \sigma^r) = 1 + n(\tau, \sigma^r)$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces Σ_τ , namely how these surfaces are packed in the 3+1 splitting.

The gauge fixing to the extra 6 primary constraints fixes the tetrads (i.e. the spatial gyroscopes and their transport law). The 4 gauge fixings to the secondary constraints (the super-Hamiltonian and super-momentum ones) fix 3K , i.e. the simultaneity 3-surface, and the 3-coordinates on it. The preservation in time of these 4 gauge fixings generates other 4 gauge fixing constraints determining the lapse and shift functions consistently with the shape of the simultaneity 3-surface and with the choice of 3-coordinates on it (here is the main difference with most of the approaches to numerical gravity).

To understand better the Hamiltonian distinction between inertial and tidal effects, i.e. the nature of the general relativistic effects, a detailed study of the Post-Newtonian solutions of Einstein's equations adopted by the IAU conventions [15] for the barycentric and geocentric celestial reference frames has begun [16]. This is no more an academic research, because in a few years the European Space Agency (ESA) will start the mission ACES [17] about the synchronization of a high-precision laser-cooled atomic clock on the space station with similar clocks on the Earth surface by means of microwave signals. If the accuracy of 5 picosec. will be achieved, it will be possible to make a coordinate-dependent test of effects at the order $1/c^3$, like the second order Sagnac effect (sensible to Earth rotational acceleration) and the general relativistic Shapiro time-delay created by the geoid [18]. It will be important to find the Post-Newtonian deviation from Einstein's convention to be able to synchronize two such clocks and to understand which metrological protocols have to be used for time dissemination at this level of accuracy. See also the Einstein Gravity Explorer proposal for a mission in the framework of ESA Cosmic Vision. The main objective of these missions

will be the high-precision measurement of the gravitational redshift around the Earth.

Gravitation in the solar system is described in the barycentric (BCRS) and geocentric (GCRS) non-rotating celestial reference systems by means of Post-Newtonian solutions of Einstein's equations for the metric tensor in harmonic gauges codified in the conventions IAU2000 [15]. In these conventions it is shown which is the relativistic structure of Newton (order $1/c^2$) and gravitomagnetic (order $1/c^3$) potentials to be used either near the Earth or in the solar system. Near the Earth the relativistic Newton potential is given mainly by the multipolar expansion of the geopotential determined by relativistic geodesy.

In space missions near the Earth the orbit of the satellite is evaluated in the GCRS at the level of few cm by using the relativistic Newton potential near the Earth and the Einstein-Infeld-Hoffmann equations at the order $1/c^2$ [4]. Instead the trajectories of the clocks in the ground stations in the GCRS are evaluated from their positions fixed on the Earth crust (ITRS, International Terrestrial Reference System) by using the non-relativistic IERS2003 conventions [19].

For the propagation of rays of light (radar signals) between the satellite and the Earth stations one uses the null geodesics of the Post-Newtonian solution in the GCRS. Till now these geodesics have been theoretically evaluated [15] only for the monopole approximation (GM/R) of the geopotential at the order $1/c^2$ in ct (i.e. with only the relativistic Newton potential) and for an axisymmetric body at the order $1/c^3$ in ct (i.e. with Newton and gravitomagnetic potentials developed in multipolar expansions) [20]. In both cases the potentials are time independent. Probably further theoretical calculations including the time-dependence of the potentials will be needed and are under investigation [16].

Let $x^\mu = (ct; \vec{x})$ be geocentric 4-coordinates with t the geocentric time. At the order $1/c^2$ the post-Newtonian solution in these 4-coordinates is $g_{00} = 1 - \frac{2U}{c^2}$, $g_{0i} = 0$, $g_{ij} = -(1 + \frac{2U}{c^2}) \delta_{ij}$ (with the convention $(+---)$ for the metric signature). The instantaneous (non-euclidean at the order $1/c^2$) 3-space is defined by $x^0 = ct = \text{const.}$ The world-line of an Earth station B is parametrized as $x_B^\mu(t) = (x_B^0 = ct; \vec{x}_B(t) = r_B(t) \hat{x}_B(t))$ ($r_B = |\vec{x}_B|$), while the world-line of the satellite A is $x_A^\mu(t) = (x_A^0 = ct; \vec{x}_A(t) = r_A(t) \hat{x}_A(t))$. Let $\vec{v}_C(t) = \frac{d\vec{x}_C(t)}{dt}$ and $\vec{a}_C(t) = \frac{d\vec{v}_C(t)}{dt}$ be the 3-velocity and 3-acceleration of $C = A, B$.

If at $t = t_A$ the satellite A emits an electro-magnetic signal, its reception at the Earth station B will happen at time $t_B > t_A$ such that $\Delta_{AB}^2 = (x_A - x_B)^2 = [c^2(t_A - t_B)^2 - \vec{N}_{AB}^2] = 0$ with $\vec{N}_{AB}(t_A, t_B) = \vec{x}_A(t_A) - \vec{x}_B(t_B) \stackrel{\text{def}}{=} R_{AB}(t_A, t_B) \hat{N}_{AB}(t_A, t_B)$, $\hat{N}_{AB}^2 = 1$.

Since in real experiments the position $\vec{x}_B(t_A)$ of the Earth station at the emission time is better known than the position $\vec{x}_B(t_B)$ at the reception time, the quantity R_{AB} has to be re-expressed in terms of the *instantaneous* distance $\vec{D}_{AB} = \vec{x}_A(t_A) - \vec{x}_B(t_A)$, $D_{AB} = |\vec{D}_{AB}|$. To order c^{-2} in ct (i.e. c^{-3} in t) one gets [15]

$$\begin{aligned}
R_{AB} &= |\vec{x}_A(t_A) - \vec{x}_B(t_B)| = \sqrt{\left[\vec{D}_{AB} + \vec{v}_B(t_A) R_{AB} + \frac{1}{2} \vec{a}_B(t_A) R_{AB}^2 + O(R_{AB}^3) \right]^2} = \\
&= D_{AB} + \frac{1}{c} \vec{D}_{AB} \cdot \vec{v}_B(t_A) + \\
&+ \frac{1}{c^2} D_{AB} \left[\vec{v}_B^2(t_A) + \frac{(\vec{D}_{AB} \cdot \vec{v}_B(t_A))}{D_{AB}^2} + \vec{D}_{AB} \cdot \vec{a}_B(t_A) \right] + O\left(\frac{1}{c^3}\right). \tag{0.1}
\end{aligned}$$

For ACES the satellite is the Space Station at the altitude of $4 \cdot 10^5 m$. The relevant quantities are: $v_A = 7.7 \cdot 10^3 m/s$, $v_B = v_{ground} = 465 m/s$, $U(\vec{x}_B)/c^2 = 6.9 \cdot 10^{-10}$, $U(\vec{x}_A)/c^2 = 6.5 \cdot 10^{-10}$. The Earth parameters are $G M_E = 3.98 \cdot 10^{14} m^3/s^2$, $R_E = 6.37 \cdot 10^6 m$. The experimental uncertainties expected for ACES are at the level of $5 ps$ for time transfer and $5 \cdot 10^{-17}$ for frequency transfer. In the one-way time and frequency transfers one must add atmosphere-dependent corrections, which tend to be compensated in the two-way transfers.

The main results of Ref.[15] are

A) *Gravitational red shift.*

A1) One-way frequency transfer

$$\begin{aligned} \frac{\nu_A}{\nu_B} &= \frac{1 - \frac{1}{c^2} \left[U(\vec{x}_B(t_B)) + \frac{1}{2} \vec{v}_B^2(t_B) \right]}{1 - \frac{1}{c^2} \left[U(\vec{x}_A(t_A)) + \frac{1}{2} \vec{v}_A^2(t_A) \right]} \frac{q_A}{q_B}, \\ q_A &= 1 - \frac{\vec{N}_{AB} \cdot \vec{v}_A(t_A)}{c} - \frac{4 G M_E}{c^3} \frac{\left(r_A(t_A) + r_B(t_B) \right) \vec{N}_{AB} \cdot \vec{v}_A(t_A) + R_{AB} \frac{\vec{x}_A(t_A) \cdot \vec{v}_A(t_A)}{r_A(t_A)}}{\left(r_A(t_A) - r_B(t_B) \right)^2 - R_{AB}^2}, \\ q_B &= 1 - \frac{\vec{N}_{AB} \cdot \vec{v}_B(t_B)}{c} - \frac{4 G M_E}{c^3} \frac{\left(r_A(t_A) + r_B(t_B) \right) \vec{N}_{AB} \cdot \vec{v}_B(t_B) + R_{AB} \frac{\vec{x}_B(t_B) \cdot \vec{v}_B(t_B)}{r_B(t_B)}}{\left(r_A(t_A) - r_B(t_B) \right)^2 - R_{AB}^2}. \end{aligned} \quad (0.2)$$

For ACES the J_2 -terms of the geopotential in the factor q_A/q_B do not exceed $4 \cdot 10^{-17}$. One has: 1) first-order Doppler effect with $|\vec{N}_{AB} \cdot \vec{v}_A/c| \leq 2.6 \cdot 10^{-5}$ for the satellite and $|\vec{N}_{AB} \cdot \vec{v}_B/c| \leq 1.6 \cdot 10^{-6}$ for the ground; 2) second order Doppler effect with $\vec{v}_A^2/2c^2 \leq 3.4 \cdot 10^{-10}$ for the satellite and $\vec{v}_B^2/2c^2 \leq 1.3 \cdot 10^{-12}$ for the ground; gravitational redshift: $U(\vec{x}_A)/c^2 = 6.5 \cdot 10^{-10}$ for the satellite and $U(\vec{x}_B)/c^2 = 6.9 \cdot 10^{-10}$ for the ground; 3) the terms of order $1/c^3$ are less than $3.6 \cdot 10^{-14}$ for the satellite and $2.2 \cdot 10^{-15}$ for the ground.

A2) Two-way frequency transfer [$U_{AB} = U(\vec{x}_B(t_B)) - U(\vec{x}_A(t_A))$, $\vec{v}_{AB} = \vec{v}_A(t_A) - \vec{v}_B(t_B)$]

$$\begin{aligned} \frac{\Delta f}{f}|_{AB} &= \frac{1}{c^2} \left(U_{AB} - \frac{1}{2} \vec{v}_{AB}^2 - \vec{N}_{AB} \cdot \vec{a}_B(t_B) \right) \left(1 + \frac{\hat{N}_{AB} \cdot \vec{v}_{AB}}{c} \right) + \\ &+ \frac{R_{AB}}{c^3} \left(-\vec{v}_A(t_A) \cdot \vec{a}_B(t_B) + \vec{N}_{AB} \cdot \frac{d\vec{a}_B(t_B)}{dt_B} + 2\vec{v}_B(t_B) \cdot \vec{a}_B(t_B) - \vec{v}_B(t_B) \cdot \vec{\partial} U_B \right). \end{aligned} \quad (0.3)$$

For ACES one has: 1) gravitational redshift: $U_{AB}/c^2 = 4.6 \cdot 10^{-11}$; second order Doppler effect: $|\vec{v}_{AB}^2/2c^2| \leq 3.3 \cdot 10^{-10}$; 3) acceleration effect: $|\vec{N}_{AB} \cdot \vec{a}_B/c^2| \leq 7 \cdot 10^{-13}$; 4) pseudo first order Doppler effect: $|\hat{N}_{AB} \cdot \vec{v}_{AB}/c| \leq 2.7 \cdot 10^{-5}$; 5) last four terms: they have maximal values $\leq 2 \cdot 10^{-17}$, $3.5 \cdot 10^{-19}$ and much less for the last two terms (negligible for ACES).

B) *Shapiro time delay.*

B1) One-way time transfer

$$\begin{aligned}
T_{AB}(t_A) &= \frac{1}{c} R_{AB} + \frac{2GM}{c^3} \ln \frac{|\vec{x}_A(t_A)| + |\vec{x}_B(t_B)| + R_{AB}}{|\vec{x}_A(t_A)| + |\vec{x}_B(t_B)| - R_{AB}} = \\
&= \frac{1}{c} D_{AB} + \frac{1}{c^2} \vec{D}_{AB} \cdot \vec{v}_B(t_A) + \frac{1}{c^3} D_{AB} \left[\vec{v}_B^2(t_A) + \frac{(\vec{D}_{AB} \cdot \vec{v}_B(t_A))^2}{D_{AB}^2} + \vec{D}_{AB} \cdot \vec{a}_B(t_A) \right] + \\
&+ \frac{2GM}{c^3} \ln \frac{|\vec{x}_A(t_A)| + |\vec{x}_B(t_A)| + D_{AB}}{|\vec{x}_A(t_A)| + |\vec{x}_B(t_A)| - D_{AB}} + O\left(\frac{1}{c^4}\right)
\end{aligned} \tag{0.4}$$

The two terms in T_{AB} beyond D_{AB}/c are usually referred to as the *Sagnac terms* of first ($1/c^2$) and second ($1/c^3$) order in t due the rotations of the Earth and the satellite. For ACES (at low elevation of the satellite) they are estimated to be of 200 ns for the first order Sagnac term, of 11 ps for the Shapiro time delay and of 5 ps for the second order Sagnac term.

B2) Two-way time transfer in the T2L2 configuration.

If we consider a signal emitted at t_A^+ by the satellite, reflected at t_B from the Earth station and re-absorbed at t_A^- by the satellite, from the two one-way time transfers $t_B - t_A^+$ and $t_A^- - t_B$ one can get the following estimate of the deviation from Einstein's 1/2 clock synchronization convention [16] (valid only when one of the two observers A and/or B is inertial)

$$\begin{aligned}
t_B &= t_A^+ + \mathcal{E} (t_A^- - t_A^+) = t_A^+ + \mathcal{E} \Delta, \\
\mathcal{E} &= \frac{1}{2} \left(1 + \frac{1}{c} [\vec{v}_A^+ + \frac{\Delta}{2} \vec{a}_A^+] \cdot \hat{n}^+ + \right. \\
&\quad \left. + \frac{1}{c^2} \left[(\vec{v}_A^+ \cdot \hat{n}^+)^2 - (\vec{v}_A^+)^2 - \frac{GM_E \Delta}{r_A^+} \frac{(\vec{v}_A^+ + \frac{\Delta}{2} \vec{a}_A^+) \cdot \hat{x}_B(t_b)}{\vec{x}_A(t_A^+) \cdot \hat{n}^+ + r_A^+} \right] \right).
\end{aligned} \tag{0.5}$$

Here \hat{n}^+ is the unit tangent to the null geodesic in the emission point at t_A^+ , i.e. the direction of the emitted light (to reach the Earth station one needs consistency with the orbit determination). $\hat{x}_B(t_B)$ is a unit vector at B. For ACES, where $t_A^- - t_A^+ \approx 10^{-3} s$, the $1/c$ term, coming from the non-inertiality of both A and B, is the dominating one: $\approx 10 ns$. The second order inertial and general relativistic effects are less than 0.1 ps.

As a consequence there is the possibility to make a clock synchronization taking into account at least the inertial effects due to the non-inertiality of both Earth stations and satellite.

See Ref.[20] for the evaluation of the time transfer and the frequency shift at the order $1/c^4$ in t beyond the monopole approximation in the case of a stationary post-newtonian gravitational field generated by an axisymmetric rotating body (the Earth with constant angular velocity $\vec{\omega}$), taking also into account the γ and β PPN parameters. For Aces, with time-keeping accuracy 10^{-18} , there is the evaluation of the contributions coming from special (Doppler effect) and general (the first four multipoles of the Earth) to the gravitational

redshift. Instead the gravito-magnetic effects coming from the intrinsic angular momentum of the Earth are negligible.

Therefore, the problem of light propagation and clock synchronization is becoming every day more important due to GPS and GALILEO [21], to the ACES mission [17], to the Bepi-Colombo mission [22] to Mercury, to the NASA LATOR proposal [23], to GAIA [24], to relativistic geodesy [25] (determination of the deviation of the geopotential from the one of the reference ellipsoid from the dependence upon it of the gravitational redshift) and to the future space navigation [26] inside the solar system. The accuracy of the new generation of atomic clocks [17] requires a better understanding of how to perform time dissemination and will induce deep modifications (like having the standard reference clock in space) in metrology [27].

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